# Modelling the industrial production of electric and gas utilities through the three-factor CIR model

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joint work with

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# Aim and scope

- In this work, we intend to predict changes in the industrial production of electric and gas utilities. The model accounts for several stylized facts such as the mean reversion of both the process and its volatility to a short-run mean, non-normality, autocorrelation and cluster volatility.
- We provide two theoretical results which are of particular importance both from theoretical and practical point of view. The first is the proof of existence and uniqueness of the solution to the SDEs system that describes the model. The second theoretical result is to convert, by the means of Lamperti transformations, the correlated system into an uncorrelated one.

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# A three-factor stochastic model

Let us denote by $\{S_t\}_{t\geq 0}$ the stochastic process modelling the level of industrial production. In addition, the correlated processes referring to the volatility and the short-run mean of $\{S_t\}_{t\geq 0}$ are, respectively, $\{v_t\}_{t\geq 0}$ and $\{\theta_t\}_{t\geq 0}$. Let $k_v, \eta, \gamma, k_\theta, \zeta, \beta, k$ and $\alpha$ be positive constants. We consider the following system of SDEs

$$\begin{cases} dS_t = k(\theta_t - S_t)dt + \alpha\sqrt{|v_t|}\sqrt{|S_t|}\, dW_t^{(1)}, \\ d\theta_t = k_\theta(\zeta - \theta_t)dt + \alpha\beta\sqrt{|v_t|}\sqrt{|\theta_t|}\, dW_t^{(2)}, \\ dv_t = k_v(\eta - v_t)dt + \gamma\sqrt{|v_t|}\, dW_t^{(3)}, \end{cases} \tag{1}$$

with the initial condition $(S_0, \theta_0, v_0) \in (0, +\infty)^3$. Here $\{W_t^{(i)}\}_{t\geq 0}$, $i = 1, 2, 3$, are three standard correlated Brownian motions such that

$$dW_t^{(1)}dW_t^{(2)} = \rho_\theta dt, \qquad dW_t^{(1)}dW_t^{(3)} = \rho_v dt, \qquad dW_t^{(2)}dW_t^{(3)} = 0, \tag{2}$$

$\rho_\theta, \rho_v \in (-1, 1)$. Moreover, the correlation coefficients satisfy the following relation

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# A three-factor stochastic model

## Lemma

*For any $t \geq 0$, the stochastic process*

$$W_t^* = \frac{W_t^{(1)} - \rho_\theta W_t^{(2)} - \rho_v W_t^{(3)}}{\sqrt{1-\rho_\theta^2-\rho_v^2}}, \tag{3}$$

*is a standard Brownian motion, which is independent both from $W_t^{(2)}$ and $W_t^{(3)}$.*

Thanks to the above Lemma, the system (1) reads as

$$\begin{cases} dS_t = k(\theta_t - S_t)dt + \alpha\sqrt{|v_t|}\sqrt{|S_t|}\left(\sqrt{1-\rho_\theta^2-\rho_v^2}dW_t^* + \rho_\theta dW_t^{(2)} + \rho_v dW_t^{(3)}\right) \\ d\theta_t = k_\theta(\zeta - \theta_t)dt + \alpha\beta\sqrt{|v_t|}\sqrt{|\theta_t|}\, dW_t^{(2)}, \\ dv_t = k_v(\eta - v_t)dt + \gamma\sqrt{|v_t|}\, dW_t^{(3)}, \end{cases} \tag{4}$$

with $(S_0, \theta_0, v_0) \in (0, +\infty)^3$ and $\{W_t^*, W_t^{(2)}, W_t^{(3)}\}_{t\geq 0}$ a three-dimensional standard Brownian motion.

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# Main Theorem

## Theorem

*Assume the Feller's condition* $2k_v\eta \geq \gamma^2$ *holds true. Then, system* (4) *admits a (local) strong solution* $\{S_t, \theta_t, v_t\}_{t\geq 0}$ *with state-space* $[0, +\infty)^2 \times (0, +\infty)$*. Pathwise uniqueness of the solution to system* (4) *holds over* $[0, \tau]$*, where*

$$\tau = \inf\{t \geq 0 : S_t = 0 \text{ or } \theta_t = 0\}. \tag{5}$$

*The random time* $\tau$ *is such that* $\mathbb{P}(\tau > 0) = 1$ *and for all* $t < \tau$ *the process* $(S_t, \theta_t, v_t)$ *takes values in* $(0, +\infty)^3$*.*

# Sketch of the proof

i) *Weak existence of a global solution:* The coefficient matrix of the diffusion term $\Sigma(S, \theta, v)$ and the drift $b(S, \theta, v)$ associated to system (4) are given by

$$\Sigma(S,\theta,v) = \begin{pmatrix} \alpha\sqrt{1-\rho_\theta^2-\rho_v^2}\sqrt{|v|}\sqrt{|S|} & \alpha\rho_\theta\sqrt{|v|}\sqrt{|S|} & \alpha\rho_v\sqrt{|v|}\sqrt{|S|} \\ 0 & \alpha\beta\sqrt{|v|}\sqrt{|\theta|} & 0 \\ 0 & 0 & \gamma\sqrt{|v|} \end{pmatrix}$$

and

$$b(S,\theta,v) = \big(k(\theta - S), k_\theta(\zeta-\theta), k_v(\eta - v)\big)^T,$$

We prove that $\Sigma$ and $b$ are continuous and satisfy the sublinear growth conditions, so that there exists a strong solution $\{S_t, \theta_t, v_t\}_{t\geq 0}$ to system (4) for any initial condition $(S_0, \theta_0, v_0) \in \mathbb{R}^3$, which does not explode in finite time (see Ikeda & Watanabe (1986) [4, Theorems 1.1, 2.3 - 2.4]).

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# Sketch of the proof

ii) *Non-negativity of solution:* Under the Feller condition, for any initial condition $v_0 > 0$, it is known that there exists a unique strong solution to the third equation in System (4), the so-called CIR-process, which is strictly positive (see, e.g., Jeanblanc et al. (2009) [5, Sections 6.3.1])).
Then, we prove by comparison result that for any initial conditions $S_0 > 0$ and $\theta_0 > 0$, both the processes $\{S_t\}_{t\geq 0}$ and $\{\theta_t\}_{t\geq 0}$ take value in $[0,+\infty)$, according to Theorem 1.1 of Ikeda & Watanabe (1986) [4, Chapter 6]).
However, we can not apply directly such theorem because the diffusion coefficient of $\{\theta_t\}_{t\geq 0}$ depends on the process $\{v_t\}_{t\geq 0}$. Therefore, we use a localization argument and define for all $N \in \mathbb{N}$

$$\eta_N = \inf\{t \geq 0 : v_t > N\}.$$

The sequence of stopping times $\{\eta_N\}_{N\in\mathbb{N}}$ is non-decreasing and such that $\lim_{N\to+\infty} \eta_N = +\infty$. Moreover, if we consider the process solving the following SDE

$$d\theta_t^1 = -k_\theta \theta_t^1 dt + \alpha\beta\sqrt{v_t}\sqrt{|\theta_t^1|}\, dW_t^{(2)}, \quad \theta_0^1 = 0 \tag{6}$$

by the Gronwall Lemma we deduce that for all $t \geq 0$, $N \in \mathbb{N}$, $\mathbb{E}[\max(\theta^1_{t\wedge\eta_N} - \theta_{t\wedge\eta_N}, 0)] = 0$, which in turn implies that $\theta_t \geq \theta_t^1 = 0, \quad \mathbb{P}-a.s. \quad \forall t \geq 0$ when $N \to +\infty$.
Analogous arguments about the process solving the following SDE

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iii) *Pathwise uniqueness of a local solution:* Notice that $\Sigma(S,\theta,v)$ is not Lipschitz-continuous in $[0,+\infty)^2 \times (0,+\infty)$ but, it is in the open set $U_N = (\frac{1}{N}, N)^3$, for any $N > 0$. By Lagrange Theorem, we can prove that $\forall (S,\theta,v),(S',\theta',v') \in U_N$ it holds

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for some constant $K_N > 0$. Then, Theorem 3.7 in Ethier & Kurtz (1986) [2], ensures the pathwise uniqueness holds over the stochastic interval $[0,\tau_N]$ where

$$\tau_N = \inf\{t \geq 0 : (S_t,\theta_t,v_t) \notin U_N\}. \quad (9)$$

Observe that $U_N \subset U_{N+1}$ hence $\tau_N$ is an increasing sequence of stopping times such that $\lim_{N\to+\infty} \tau_N = \tau$, because $\{S_t\}_{t\geq0}$, $\{\theta_t\}_{t\geq0}$ and $\{v_t\}_{t\geq0}$ do not explode and $v_t > 0$, $\forall t \geq 0$. Thus, by letting $N \to +\infty$ we obtain pathwise uniqueness in $[0,\tau]$.

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# Lamperti transformation

Thanks to the strictly positivity of $(S_t, \theta_t, v_t)$ for all $t < \tau$ we will apply a suitable Lamperti transformation which converts the correlated system (4) into an uncorrelated one. We give this result in a more general framework, and we consider the following system of SDEs

$$\begin{cases} dS_t = \mu(S_t, \theta_t, v_t)dt + \sigma(v_t)\Gamma(S_t)(\sqrt{1-\rho_\theta^2-\rho_v^2}\, dW_t^* + \rho_\theta dW_t^{(2)} + \rho_v dW_t^{(3)}) \\ d\theta_t = \mu_\theta(\theta_t)dt + \sigma(v_t)\sigma_\theta(\theta_t)dW_t^{(2)} \\ dv_t = \mu_v(v_t)dt + \sigma_v(v_t)dW_t^{(3)}, \end{cases} \tag{10}$$

with $(S_0, \theta_0, v_0) \in (0, +\infty)^3$. Let us assume the functions $\sigma(x)$, $\Gamma(x)$, $\sigma_\theta(x)$, $\sigma_v(x)$ are strictly positive and continuously differentiable for $x \in (0, +\infty)$, $\frac{1}{\Gamma(x)}$, $\frac{1}{\sigma_\theta(x)}$, and $\frac{\sigma(x)}{\sigma_v(x)}$ are integrable on $(0, +\infty)$. With the choice

$$\mu_v(v) = k_v(\eta - v), \qquad \sigma_v(v) = \gamma\sqrt{|v|}, \tag{11}$$

$$\mu_\theta(\theta) = k_\theta(\zeta - \theta), \qquad \sigma_\theta(v) = \beta\sqrt{|\theta|}, \tag{12}$$

$$\mu(S, \theta, v) = k(\theta - S), \qquad \sigma(v) = \alpha\sqrt{|v|}, \qquad \Gamma(S) = \sqrt{|S|}; \tag{13}$$

system (10) reduces to (4).

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with $(S_0, \theta_0, v_0) \in (0, +\infty)^3$. Let us assume the functions $\sigma(x)$, $\Gamma(x)$, $\sigma_\theta(x)$, $\sigma_v(x)$ are strictly positive and continuously differentiable for $x \in (0, +\infty)$, $\frac{1}{\Gamma(x)}$, $\frac{1}{\sigma_\theta(x)}$, and $\frac{\sigma(x)}{\sigma_v(x)}$ are integrable on $(0, +\infty)$. With the choice

$$\mu_v(v) = k_v(\eta - v), \qquad \sigma_v(v) = \gamma\sqrt{|v|}, \tag{11}$$

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system (10) reduces to (4).

# Auxiliary process

Let us introduce the auxiliary process

$$X_t = g(S_t) - \rho_\theta l(\theta_t) - \rho_v f(v_t), \quad t \in [0, \tau] \tag{14}$$

## Proposition

*The system of SDE's in* (1) *is equivalent to the following*

$$\begin{cases} dX_t = \left( \frac{k\theta_t}{\sqrt{\varphi(X_t, \theta_t, v_t)}} - k\sqrt{\varphi(X_t, \theta_t, v_t)} - \sum_{u=0}^{2} c_{u,t} \right) dt + \alpha\sqrt{v_t}\sqrt{1 - \rho_\theta^2 - \rho_v^2}\, dW_t^* \\ d\theta_t = k_\theta(\zeta - \theta_t)dt + \alpha\beta\sqrt{v_t}\sqrt{\theta_t}\, dW_t^{(2)} \\ dv_t = k_v(\eta - v_t)dt + \gamma\sqrt{v_t}\, dW_t^{(3)}, \end{cases} \tag{15}$$

*with initial condition* $(X_0 = 2\sqrt{S_0} - \frac{2\rho_\theta}{\beta}\sqrt{\theta_0} - \frac{\rho_v \alpha}{\gamma} v_0, \theta_0, v_0)$ *and*

$$c_t = \frac{2\rho_\theta}{\beta}\sqrt{\theta_t} + \frac{\rho_v \alpha}{\gamma} v_t, \qquad \varphi(X_t, \theta_t, v_t) = \left( \frac{X_t + c_t}{2} \right)^2, \tag{16}$$

$$c_{0,t} = \frac{\alpha^2 v_t}{4\sqrt{\varphi(X_t, \theta_t, v_t)}}, \qquad c_{1,t} = \rho_\theta \left( \frac{k_\theta(\zeta - \theta_t)}{\beta\sqrt{\theta_t}} - \frac{\beta\alpha^2 v_t}{4\sqrt{\theta_t}} \right), \qquad c_{2,t} = \frac{\rho_v \alpha k_v(\eta - v_t)}{\gamma}. \tag{17}$$

# Dataset

- Figure 1 displays the percent change in the industrial production of electric and gas utilities IPUTIL, as classified by the North American Industry Classification System (NAICS).

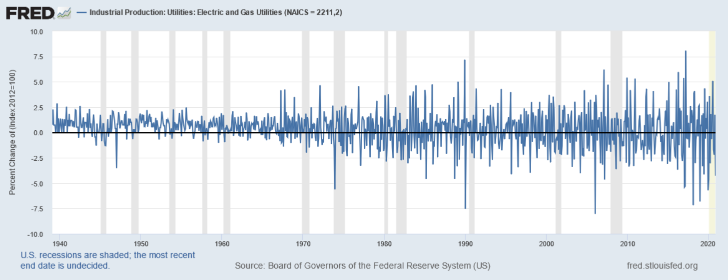


Figure: Board of Governors of the Federal Reserve System (US), Industrial Production: Electric and Gas Utilities [3]. Percent change. Data from 1939-02-01 to 2021-11-01. Shaded grey areas correspond to recessions and the yellow strip to the right highlights the COVID-19 pandemic.

# Numerical simulations

- Let $(s_1, ..., s_n)$ be the observations of $S_t$, and $(\Theta_1, ..., \Theta_n)$ those of the mean process $\theta_t$, taken as the exponential weighted moving average (EWMA) of $(s_1, ..., s_n)$. Moreover, the observations $(\nu_1, ..., \nu_n)$ of the volatility process $v_t$ are given by the so-called *pointwise volatility* $\nu_u = |s_u - \Theta_u|$ $(1 \leq u \leq n)$.
- In order to calibrate the model parameters, we consider the estimating function approach for ergodic diffusion models introduced in Bibby et al. (2010) [1].
- To simulate the processes $v_t, \theta_t$ we apply the strong convergent Milstein discretization (see Mil'shtein (1979) [6]) to the second and third SDE of Eq. (1). Hence, for any $1 \leq u \leq (n-1)$ we compute

$$\hat{v}_{u+1} = \hat{v}_u + \hat{k}_v(\hat{\eta} - \hat{v}_u)\,\Delta + \hat{\gamma}\sqrt{\hat{v}_u\Delta}\,\varepsilon^{(3)}_{u+1} + \frac{\hat{\gamma}^2}{4}[(\sqrt{\Delta}\,\varepsilon^{(3)}_{u+1})^2 - \Delta], \quad (18)$$

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# In sample simulation

- Once calibrated the model parameters, we simulate the auxiliary process $X_t$

$$\hat{X}_{u+1} = \hat{X}_u + \omega(\hat{X}_u, \hat{\theta}_u, \hat{v}_u) + \hat{\alpha}\sqrt{\hat{v}_u \Delta}\sqrt{1 - \hat{\rho}_\theta^2 - \hat{\rho}_v^2}\, \varepsilon_{u+1}^{(1)}, \qquad (20)$$

and then obtain $\boxed{\hat{S}_{u+1} = g^{-1}(\hat{X}_{u+1} + \hat{\rho}_\theta l(\hat{\theta}_{u+1}) + \hat{\rho}_v f(\hat{v}_{u+1})).}$

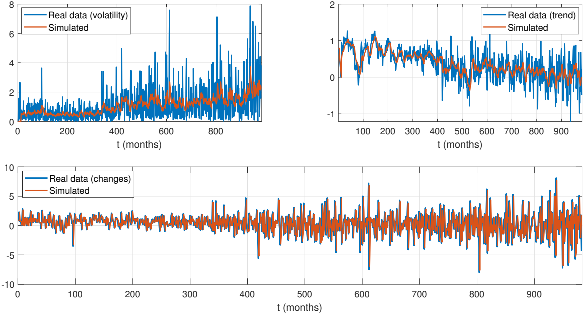


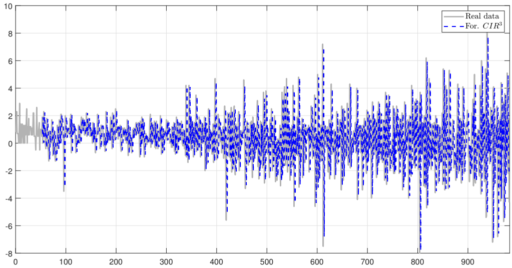

Figure: Real data (changes) versus simulated data via the $CIR^3$ model Eq. (1). The top left graph shows the volatility, while the top right graph shows the trend (i.e., the EWMA). The bottom graph in the center displays the changes of real data.

# Forecasting

To predict changes in the industrial production of electric and gas utilities through our model in system (1), we take the Monte Carlo approximation, i.e.

$$\hat{X}_{u+z} = \frac{1}{N} \sum_{r=1}^{N} \hat{X}_{u+z,r} \qquad (z \geq 1), \tag{21}$$

where, for each iteration $r$, $\hat{X}_{u+z,r}$ is computed as in Eq. (20), and $N = 100,000$.




| Err. Measure | Forecast Hor. | Proposed Model | ARIMA-GARCH | NRM |
|---|---|---|---|---|
| MAPE | 1 Month | 0.1092 | 0.1629 | 0.1594 |
| NRMSE | 1 Month | 0.0575 | 0.0853 | 0.0820 |

Table: MAPE and NRMSE obtained over the horizon of 1 month. Rolling window set to 36 months. Out of sample results.

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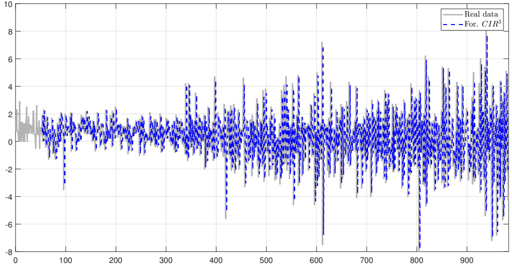


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Thank you for your attention!

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